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A Wick Rotation for Spinor Fields: the Canonical Approach

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ABSTRACT

Recently we proposed a new Wick rotation for Dirac spinors which resulted in a hermitean action in Euclidean space. Our work was in a path integral context, however, in this note, we provide the canonical formulation of the new Wick rotation along the lines of the proposal of Osterwalder and Schrader.

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1 Introduction and Review.

Field theories Wick rotated to Euclidean space are the subject of an enormous body of research. In particular, modern non-perturbative studies of supersymmetric theories (for example instantons and the study of Donaldson invariants for compact manifolds) depend on the introduction of Euclidean field theories. Clearly then, it is of crucial importance to understand how one performs a Wick rotation for spinors.

In a previous publication in this journal [1] we observed that there existed two apparently distinct approaches to Dirac spinors in Euclidean space. Namely the approach of Osterwalder and Schrader [3] (OS) in which the fields ψ and its Dirac conjugate $\bar{\psi}$ are taken to be independent and hermiticity is forsaken and the approach of Schwinger [4] and later Zumino [5] in which spinor degrees of freedom are undoubled and the action in Euclidean space is hermitean. Within a path integral context, the distinction between integrating over fields ψ and ψ^\dagger versus independent fields ψ and χ^\dagger ($\psi \neq \chi$) is only semantic due to the algebraic nature of Grassmann integration, so the real puzzle was therefore to understand how Schwinger was able to maintain hermiticity whereas OS did not. This problem was solved by introducing a new Wick rotation for Dirac spinors which acted only on the fundamental fields and coordinates.

For vectors $A_\mu(t, \vec{x})$, the Wick rotation to Euclidean space is performed by transforming both the time coordinate $t \rightarrow -i\tau$ and vector indices by a matrix $\Omega_\mu{}^\nu = \text{diag}(i, 1, 1, 1)$, i.e., $A_\mu(t, \vec{x}) \rightarrow \Omega_\mu{}^\nu A_\mu^E(\tau, \vec{x})$. However for complex vectors the complex conjugate A_μ^\dagger transforms under the same matrix $\Omega_\mu{}^\nu$ rather than $\Omega_\mu{}^{\nu\dagger}$. This observation led us to introduce the following Wick rotation for Dirac spinors [1]:

$$\psi(t, \vec{x}) \rightarrow S(\theta)\psi_\theta(t_\theta, \vec{x}) \quad (1)$$

$$\psi^\dagger(t, \vec{x}) \rightarrow \psi_\theta^\dagger(t_\theta, \vec{x})S(\theta) \quad (2)$$

$$t \rightarrow e^{-i\theta}t_\theta, \quad (3)$$

where²

$$S(\theta) = e^{\gamma^4\gamma^5\theta/2}, \quad \theta \in [0, \pi/2]. \quad (4)$$

²Our conventions are as follows, the Minkowski Dirac matrices $\gamma^\mu = (\gamma^0 \equiv -i\gamma^4, \vec{\gamma})^\mu$, where γ^4 and $\vec{\gamma}$ are hermitean and $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2\text{diag}(-1, 1, 1, 1)^{\mu\nu}$. The matrix $\gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4$ is hermitean.

The matrix $S(\theta)$ is unitary and the parameter θ is introduced to provide a continuous³ interpolation between the Minkowski and Euclidean theories. At the initial value $\theta = 0$, $S(\theta = 0) = I$ and $\psi_{\theta=0} \equiv \psi$, $\psi_{\theta=0}^\dagger \equiv \psi^\dagger$ and $t_{\theta=0} \equiv t \equiv x^0 \equiv -x_0$ take their usual Minkowski values, whereas at the endpoint $\theta = \pi/2$, $S(\theta = \pi/2) = e^{\gamma^4 \gamma^5 \pi/4} \equiv S$ and $\psi_{\theta=\pi/2} \equiv \psi_E$, $\psi_{\theta=\pi/2}^\dagger \equiv \psi_E^\dagger$ and $t_{\theta=\pi/2} \equiv \tau \equiv x^4 \equiv x_4$ are their Euclidean counterparts. Observe that $S(\theta)\gamma^4 = \gamma^4 S^{-1}(\theta)$ whereby

$$\bar{\psi}(t, \vec{x}) \equiv \psi^\dagger(t, \vec{x})\gamma^4 \rightarrow \psi_\theta^\dagger(t_\theta, \vec{x})\gamma^4 S^{-1}(\theta), \quad (5)$$

so that our Wick rotation induces a similarity transformation upon the Dirac matrices in Dirac bilinears $\bar{\psi} \Gamma_A \psi$ for some combination of Dirac matrices Γ_A ,

$$\vec{\gamma}(\theta) \equiv S^{-1}(\theta)\vec{\gamma}S(\theta) = \vec{\gamma} \equiv \vec{\gamma}_E \quad (6)$$

$$\gamma^4(\theta) \equiv S^{-1}(\theta)\gamma^4 S(\theta) = \gamma^4 \cos \theta + \gamma^5 \sin \theta \quad (7)$$

$$\gamma^5(\theta) \equiv S^{-1}(\theta)\gamma^5 S(\theta) = -\gamma^4 \sin \theta + \gamma^5 \cos \theta. \quad (8)$$

Note then that $\gamma_{\theta=\pi/2}^4 \equiv \gamma_E^4 = \gamma^5$, $\gamma_{\theta=\pi/2}^5 \equiv \gamma_E^5 = -\gamma^4$ whereby the Wick rotation of the Dirac conjugate spinor $\bar{\psi}$ yields

$$\bar{\psi}(t, \vec{x}) = \psi^\dagger(t, \vec{x})\gamma^4 \rightarrow -\psi_E^\dagger(\tau, \vec{x})\gamma_E^5 e^{-\gamma^4 \gamma^5 \theta/4} \quad (9)$$

and the troublesome γ^4 in $\bar{\psi}$ has been reinterpreted as $-\gamma_E^5$ in the Euclidean theory. Applying this Wick rotation to the action for a free massive Dirac spinor

$$\frac{i}{\hbar} S_M \equiv -\frac{1}{\hbar} \int dt d\vec{x} \psi^\dagger(t, \vec{x}) \gamma^4 (\gamma^0 \frac{\partial}{\partial t} + \vec{\gamma} \cdot \vec{\partial} + m) \psi(t, \vec{x}), \quad (10)$$

we obtain

$$-\frac{1}{\hbar} S_E \equiv \frac{1}{\hbar} \int d\tau d\vec{x} \psi_E^\dagger(\tau, \vec{x}) \gamma_E^5 (\gamma^4 \frac{\partial}{\partial \tau} + \vec{\gamma} \cdot \vec{\partial} + m) \psi(\tau, \vec{x}). \quad (11)$$

This action is hermitean, $SO(4)$ invariant and is the result of a Wick rotation acting as an analytic continuation $t \rightarrow -i\tau$ and a *simultaneous rotation on spinor indices*.

³Note that Lorentz invariance of the intermediate interpolating theories is obtained only if one complexifies so that ψ_θ and ψ_θ^\dagger are independent spinors and hermiticity of the interpolating Dirac action is lost. Our claim is not that ψ and ψ^\dagger remain dependent under Wick rotation, but rather, that for Dirac spinors, hermiticity at the endpoint (the Euclidean theory) may be regained. In this sense one may think of hermiticity as a symmetry property of the Euclidean theory.

The above Wick rotation produces a Euclidean theory for spinor fields whose action, in the exponent of a Euclidean path integral, yields Euclidean Greens functions which are related to the usual Minkowski Greens functions by analytic continuation and a rotation on spinor indices by the matrix S introduced above. However, along with necessary and sufficient conditions for a Euclidean theory to produce the analytically continued counterparts of the Greens function of a given Minkowski theory [2], Osterwalder and Schrader [3] have explicitly constructed a canonical Euclidean theory in terms of Euclidean Dirac spinor fields acting in a Euclidean Fock space whose Greens functions are the analytic continuations of the corresponding Minkowski Greens functions. In the remainder of this note, we shall reformulate their work in the context of our Wick rotation for Dirac spinors.

2 The Main Ingredients of the OS construction.

We begin by briefly sketching the main ingredients of the OS construction. Firstly we analytically continue canonical Minkowski fields to imaginary times $t \rightarrow -i\tau$. The canonical (Heisenberg) Minkowski field satisfying the free Dirac equation $(\not{\partial} + m)\psi = 0$ is given by

$$\psi(t, \vec{x}) = e^{iH_0 t} \psi(0, \vec{x}) e^{-iH_0 t}, \quad (12)$$

with the usual mode expansion

$$\psi(0, \vec{x}) = \int \frac{d\vec{k}}{(2\pi)^{3/2}} \left(b_{\vec{k}} \cdot u_{\vec{k}} e^{i\vec{k} \cdot \vec{x}} + d_{\vec{k}}^\dagger \cdot v_{-\vec{k}} e^{-i\vec{k} \cdot \vec{x}} \right) \quad (13)$$

and normal ordered free Hamiltonian $H_0 = \int d\vec{k} \omega_{\vec{k}} (b_{\vec{k}}^\dagger \cdot b_{\vec{k}} + d_{\vec{k}}^\dagger \cdot d_{\vec{k}})$ where $\omega_{\vec{k}} = (\vec{k}^2 + m^2)^{1/2}$. We denote the sum over spin polarizations by a dot, i.e., $b_{\vec{k}} \cdot u_{\vec{k}} = \sum_{r=1,2} b_{\vec{k}}^r \cdot u_{\vec{k}}^r$. The orthonormal spinor wave functions satisfy $(i\not{k} + m)u_{\vec{k}} = 0 = (-i\not{k} + m)v_{-\vec{k}}$ and spin polarization sums

$$u_{\vec{k}} \cdot \bar{u}_{\vec{k}} = \frac{-i\not{k} + m}{2\omega_{\vec{k}}} ; \quad v_{-\vec{k}} \cdot \bar{v}_{-\vec{k}} = \frac{-i\not{k} - m}{2\omega_{\vec{k}}}. \quad (14)$$

Defining the usual Minkowski vacuum $|0\rangle$ via $d_{\vec{k}}|0\rangle = 0 = b_{\vec{k}}|0\rangle$ and imposing commutation relations for the modes

$$\{b_{\vec{k}}, b_{\vec{k}'}^\dagger\} = \delta^3(\vec{k} - \vec{k}')1 = \{d_{\vec{k}}, d_{\vec{k}'}^\dagger\} \quad (15)$$

(we have suppressed the polarization indices $r, s = \pm$ so 1 denotes δ^{rs}) one obtains the two-point function (propagator)

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = -i \int \frac{dk_0 d^3k}{(2\pi)^4} \frac{-i\not{k} + m}{k^2 + m^2 - i\epsilon} e^{ik_\mu(x-y)^\mu} \equiv \mathcal{D}(x-y). \quad (16)$$

The analytically continued Minkowski fields are constructed by allowing the Minkowski field (12) to undergo imaginary time evolution with $t = -i\tau$,

$$\begin{aligned} \psi(-i\tau, \vec{x}) &= e^{H_0\tau} \psi(0, \vec{x}) e^{-H_0\tau} \\ &= \int \frac{d^3k}{\sqrt{(2\pi)^3}} \left\{ b_{\vec{k}} \cdot u_{\vec{k}} e^{-\omega_{\vec{k}}\tau + i\vec{k} \cdot \vec{x}} + d_{\vec{k}}^\dagger \cdot v_{-\vec{k}} e^{\omega_{\vec{k}}\tau - i\vec{k} \cdot \vec{x}} \right\} \end{aligned} \quad (17)$$

The continuation of the Dirac conjugate field $\bar{\psi}$ is defined in the same way

$$\begin{aligned} \bar{\psi}(-i\tau, \vec{x}) &= e^{H_0\tau} \psi^\dagger(0, \vec{x}) i\gamma^0 e^{-H_0\tau} \\ &= \int \frac{d^3k}{\sqrt{(2\pi)^3}} \left\{ b_{\vec{k}}^\dagger \cdot \bar{u}_{\vec{k}} e^{+\omega_{\vec{k}}\tau - i\vec{k} \cdot \vec{x}} + d_{\vec{k}} \cdot \bar{v}_{-\vec{k}} e^{-\omega_{\vec{k}}\tau + i\vec{k} \cdot \vec{x}} \right\} \\ &= [\psi(\vec{x}, +i\tau)]^\dagger i\gamma^0. \end{aligned} \quad (18)$$

It is important to realize from the last equality in (18) that hermitean conjugation and analytic continuation do not commute, but rather an additional “Euclidean time reversal” $\tau \rightarrow -\tau$ is required. The concept of reflection positivity follows from this remark. The continued free two-point function, or propagator, can now straightforwardly be constructed from the continued Minkowski fields,

$$\begin{aligned} \mathcal{D}(-i\tau + i\sigma, \vec{x} - \vec{y}) &\equiv \langle 0|\tilde{T}\psi(-i\tau, \vec{x})\bar{\psi}(-i\sigma, \vec{y})|0\rangle \\ &= \theta(\tau - \sigma) \langle 0|\psi(-i\tau, \vec{x})\bar{\psi}(-i\sigma, \vec{y})|0\rangle \\ &\quad - \theta(\sigma - \tau) \langle 0|\bar{\psi}(-i\sigma, \vec{y})\psi(-i\tau, \vec{x})|0\rangle \\ &= \int \frac{dk_4 d^3k}{(2\pi)^4} \frac{-i\not{k}^{\text{OS}} + m}{k_{\text{E}}^2 + m^2} e^{ik_{\mu}^{\text{E}}(x-y)_{\mu}^{\text{E}}} \equiv \mathcal{D}_{\text{E}}(x^{\text{E}} - y^{\text{E}}). \end{aligned} \quad (19)$$

Let us make a few comments. The symbol \tilde{T} denotes time ordering, but now with respect to τ , the continued time. Further, $k^{\text{OS}} \equiv \gamma^4 k_4 + \vec{\gamma} \cdot \vec{k}$ so that *spinor indices undergo no rotation* in the OS approach, rather an i is simply “borrowed” from the relation $\gamma^0 = -i\gamma^4$. Also $k_\mu^{\text{E}} x_\mu^{\text{E}} \equiv k_4 \tau + \vec{k} \cdot \vec{x}$ is the Euclidean inner product. Furthermore, it is easy to show that this result for $\mathcal{D}(-i\tau, \vec{x})$ is exactly the same as that obtained from the function $\mathcal{D}(t, \vec{x})$ in (16) by a direct (unique) analytic continuation in the time variable.

So far we have done nothing except consider the usual Minkowski fields acting in the Minkowski Fock space but at imaginary values of the time coordinate. However in the last line of (19) we denoted $\mathcal{D}(-i\tau, \vec{x}) = \mathcal{D}_{\text{E}}(x^{\text{E}})$ because the next step is to construct Euclidean fields acting in a Euclidean Fock space whose two-point function is $\mathcal{D}_{\text{E}}(x^{\text{E}})$. General Greens functions can be reconstructed via use of the Wick theorem.

The time coordinate should play no preferred rôle in Euclidean space so the OS fields anticommute at all points x^{E} and y^{E} (x^{E} and y^{E} are now, of course, four dimensional). Hence there is no time ordering in their Euclidean formulation and the Euclidean fields are expanded in terms of modes depending on Euclidean four-momenta. In any case there are no plane wave solutions to $(-\square_{\text{E}} + m^2)\psi_{\text{E}} = 0$ in flat Euclidean space, so that, in the OS construction, Euclidean fields are off-shell. Hence their Euclidean fields have a form which is rather close to that of a five-dimensional Minkowski field at zero time. Explicitly the Euclidean canonical Dirac spinor of OS is given by

$$\psi_{\text{E}}(x) = \int \frac{d^4 k}{\sqrt{(2\pi)^4 \Omega_k}} \left\{ B_k \circ U_k e^{ikx} + D_k^\dagger \circ V_{-k} e^{-ikx} \right\}, \quad (20)$$

where $\Omega_k \equiv (k^2 + m^2)^{1/2}$, $e^{ikx} = e^{i(k_4 \tau + \vec{k} \cdot \vec{x})}$ and from now on we drop the subscript E on the Euclidean four-vectors k and x . The mode operators B_k and D_k^\dagger act in a Euclidean Fock space with vacuum $|0\rangle_{\text{E}}$ defined such that $B_k|0\rangle_{\text{E}} = 0 = D_k|0\rangle_{\text{E}}$. The only non-vanishing anticommutation relations of the modes now give a four dimensional delta function,

$$\{B_k, B_{k'}^\dagger\} = \delta^4(k - k')1 = \{D_k, D_{k'}^\dagger\}, \quad (21)$$

where we again suppress polarization indices so that 1 denotes δ^{RS} . However, let us stress that the indices R and S no longer run over values 1, 2, rather it is necessary to *double* the spin polarization degrees of freedom whereby

$R, S = 1, \dots, 4$. and we denote $B_k \circ U_k \equiv \sum_{R=1}^4 B_k^R U_k^R$. The spinor wave functions U_k and V_{-k} do not satisfy any equations of motion. At this point the fields $\psi_E(x)$ in (20) and $\psi(-i\tau, \vec{x})$ in (17) are totally unrelated, they act in different Fock spaces.

The next task is to construct a conjugate momentum field to $\psi_E(x)$ (i.e., the analogue of ψ^\dagger in the Minkowski case). In the OS proposal, the answer is no longer $[\psi_E(x)]^\dagger$ (although a key feature of the canonical formulation of our new Wick rotation is that this property will be retained), instead the operation of hermitean conjugation is replaced by the composition of hermitean conjugation and a unitary involution Θ defined as follows. For Bose fields $\phi_E(\tau, \vec{x})$ (which may be easily be treated in the OS approach without doubling, see [3] for details) Θ acts simply as Euclidean time reversal,

$$\Theta \phi_E(\tau, \vec{x}) \Theta^{-1} = \phi_E(-\tau, \vec{x}) = \phi_E(\Theta x), \quad (22)$$

where $\Theta(\tau, \vec{x}) = (-\tau, \vec{x})$. This definition is motivated by the remark above that for continued Minkowski fields, one needed an additional Euclidean time reversal when comparing imaginary time evolution of the hermitean conjugated field with the hermitean conjugate of the imaginary time evolved field (see (18)). For spinors however, the action of Θ is more subtle. Define a field $\chi_E^\dagger(x)$, the Euclidean analogue of the Minkowski field $\bar{\psi}(\vec{x}, t)$, by

$$\chi_E^\dagger(x) = \int \frac{d^4 k}{\sqrt{(2\pi)^4 \Omega_k}} \left\{ B_k^\dagger \circ W_k^\dagger e^{-ikx} + D_k \circ X_{-k}^\dagger e^{ikx} \right\}. \quad (23)$$

Since we already doubled the number of spin polarizations ($R, S = 1, \dots, 4$), it would be a redoubling if the field $\chi_E^\dagger(x)$ were independent⁴ of $\psi_E(x)$. Rather, χ_E^\dagger is related to ψ_E by both hermitean conjugation and the action of Θ ,

$$\chi_E^\dagger(\tau, \vec{x}) \equiv \Theta^{-1} \psi_E^\dagger(-\tau, \vec{x}) i\gamma^0 \Theta = \Theta^{-1} \psi_E^\dagger(\Theta x) i\gamma^0 \Theta. \quad (24)$$

The relation (24) together with the expansions (23) and (20) imply that Θ has a more complicated action on the modes (and in turn states in the Euclidean Fock space) such that

$$\Theta^{-1} B_{\Theta k}^\dagger \circ U_{\Theta k}^\dagger i\gamma^0 \Theta = B_k^\dagger \circ W_k^\dagger; \quad \Theta^{-1} D_{\Theta k} \circ V_{-\Theta k}^\dagger i\gamma^0 \Theta = D_k \circ X_{-k}^\dagger \quad (25)$$

⁴Contrast this to an OS path integral approach for Dirac spinors in which the “doubling” of spinor degrees of freedom in Euclidean space is introduced by taking the field χ_E^\dagger to be independent of ψ_E .

where $\Theta k = (-k^4, \vec{k})$. Given the explicit forms (see [3]) of the spinor wavefunctions U_k, V_{-k}, W_k and X_{-k} one can write down the action of Θ on the mode operators B_k and D_k . For our purposes it is enough to note that the spinor wavefunctions satisfy spin polarization sums constructed such that one obtains the desired two-point function in (19),

$$U_k \circ W_k^\dagger = \frac{-i\not{k}^{\text{OS}} + m}{\Omega_k} ; \quad V_{-k} \circ X_{-k}^\dagger = \frac{-i\not{k}^{\text{OS}} - m}{\Omega_k}. \quad (26)$$

Defining $\Theta|0\rangle_{\text{E}} = |0\rangle_{\text{E}}$ one can then also calculate the action of Θ on states in the Euclidean Fock space. The mode expansions (20) and (23) along with the spin polarization sums in (26) yield

$$\{\psi_{\text{E}}(x), \chi_{\text{E}}^\dagger(y)\} = 0. \quad (27)$$

Furthermore, using (20), (21), (23) and (26) it is easy to verify the following equalities

$$\mathcal{D}_{\text{E}}(x-y) \equiv {}_{\text{E}}\langle 0 | \psi_{\text{E}}(x) \chi_{\text{E}}^\dagger(y) | 0 \rangle_{\text{E}} = -{}_{\text{E}}\langle 0 | \chi_{\text{E}}^\dagger(y) \psi_{\text{E}}(x) | 0 \rangle_{\text{E}} = \mathcal{D}(-i\tau + i\sigma, \vec{x} - \vec{y}), \quad (28)$$

where $D_{\text{E}}(x-y)$ in (28) agrees with $D_{\text{E}}(x-y)$ in (19) so that the Euclidean two-point function without time ordering reproduces the continued, time ordered Minkowski two-point function.

The final ingredient is the relation between states in the Euclidean Fock space, and those in the physical Minkowski Hilbert space. This is provided by the following mapping

$$W : |X\rangle_{\text{E}} \rightarrow |WX\rangle_{\text{M}}, \quad (29)$$

from an arbitrary state $|X\rangle_{\text{E}}$ in the Euclidean Fock space to some state $|WX\rangle_{\text{M}}$ in the Minkowski Hilbert space. We shall call this mapping the “OS–Wick map” and it is defined as follows. A general state in the Euclidean Fock space [3] may be represented as

$$|X\rangle_{\text{E}} = \int d^4x_1 \dots d^4x_{m+n} f_1(x_1) \dots f_{m+n}(x_{m+n}) : \underbrace{\psi_{\text{E}}(x_1) \dots \chi_{\text{E}}^\dagger(x_{m+n})}_{m \text{ } \psi_{\text{E}}\text{'s, } n \text{ } \chi_{\text{E}}^\dagger\text{'s}} : |0\rangle_{\text{E}}. \quad (30)$$

The functions $f_i(x_i)$ are some choice of test functions and it is convenient to normal order this expression as denoted by $: :$ by which we mean all

annihilation operators B_k and D_k are to be pulled to the right. The OS–Wick map may now be defined by its action on the state $|X\rangle_E$ in (30),

$$W|X\rangle_E = |WX\rangle_M \equiv: \psi(-i\tau_1, \vec{x}_1) \dots \bar{\psi}(-i\tau_{m+n}, \vec{x}_{m+n}) : |0\rangle. \quad (31)$$

where, for brevity, we have suppressed the smearing by test functions $f_i(x_i)$. The fields $\psi(-i\tau_1, \vec{x}_1)$ and $\bar{\psi}(-i\tau_{m+n}, \vec{x}_{m+n})$ are precisely the continued Minkowski fields defined in (17) and (18), respectively, above. In [3], the following central theorem is proven

$${}_E\langle \Theta X | Y \rangle_E = {}_M\langle WX | WY \rangle_M. \quad (32)$$

which states that inner products of states in the Euclidean Fock space are related to those in the Minkowski Hilbert space by the OS–Wick map and the unitary involution Θ . The inner product in the Minkowski Hilbert space should be positive definite, whereby we immediately obtain the OS reflection positivity condition

$${}_E\langle \Theta X | X \rangle_E \geq 0. \quad (33)$$

As yet we have made no mention of how dynamics are included in this proposal, but at this point we refer the reader to the original work of Osterwalder and Schrader [3]. Let us now give the generalization of the above construction to include our new Wick rotation.

3 The Canonical Formulation of the New Wick Rotation.

In [3], it is argued that the field χ_E^\dagger cannot be replaced by ψ_E^\dagger since the two-point function,

$${}_E\langle 0 | \psi_E(x) \chi_E^\dagger(y) | 0 \rangle_E = \int \frac{d^4k}{(2\pi)^4} \frac{-i\cancel{k}^{\text{OS}} + m}{k_E^2 + m^2} e^{ik_E(x-y)_E}, \quad (34)$$

is then inconsistent because only the left hand side is invariant under hermitean conjugation and the interchange of x and y . In light of our new Wick rotation, the remedy is obvious. One should replace \cancel{k}^{OS} by $\cancel{k}^E = k_E^\mu \gamma_E^\mu$ where the matrices γ_E^μ are defined in the introduction (and were obtained via a similarity transformation induced by the rotation of spinor indices) and put

$$\chi_E^\dagger = -\psi_E^\dagger \gamma_E^5. \quad (35)$$

In order to incorporate our rotation of spinor indices in the OS proposal, we replace the OS–Wick map W by a new OS–Wick map \widetilde{W} , defined as follows

$$\widetilde{W} : |X\rangle \rightarrow |\widetilde{W}X\rangle_{\text{M}} \quad (36)$$

where, in general,

$$|X\rangle_{\text{E}} =: \psi_{\text{E}}(x_1) \dots \psi_{\text{E}}^{\dagger}(x_{n+m})(-\gamma_{\text{E}}^5) : |0\rangle_{\text{E}} \quad (37)$$

and

$$|\widetilde{W}X\rangle_{\text{M}} =: S\psi(-i\tau_1, \vec{x}_1) \dots \overline{\psi}(-i\tau_{n+m}, \vec{x}_{n+m})S^{-1} : |0\rangle_{\text{M}} \quad (38)$$

with $S = e^{\gamma^4 \gamma^5 \pi/4}$. Hence, Euclidean Greens functions are related to their continued Minkowski Greens functions by an additional rotation of spinor indices. For example, for the free two-point function, we find

$$\begin{aligned} {}_{\text{E}}\langle 0 | \psi_{\text{E}}(x) \psi_{\text{E}}^{\dagger}(y) (-\gamma_{\text{E}}^5) | 0 \rangle_{\text{E}} &= \int \frac{d^4 k}{(2\pi)^4} \frac{-i k^{\text{E}} + m}{k^2 + m^2} e^{ik(x-y)} \\ &= S^{-1} \mathcal{D}(-i\tau + i\sigma, \vec{x} - \vec{y}) S. \end{aligned} \quad (39)$$

We must now construct the field ψ_{E} , the Euclidean Fock space and the unitary involution Θ . Let us briefly mention two unfruitful avenues before we give our solution. The first would be to require W_k^{\dagger} in (23) and U_k in (20) to satisfy

$$W_k^{\dagger} = -U_k^{\dagger} \gamma_{\text{E}}^5. \quad (40)$$

(along with an analogous condition on V_{-k} and X_{-k}). However, we must still reproduce the correct continued Minkowski two-point function which requires $U_k \circ W_k^{\dagger} = -i k^{\text{E}} + m$ for which there is no solution when (40) holds⁵.

A second, bolder proposal would be to notice that the expressions (20) and (23) are reminiscent of those of a five ($5=1+4$) dimensional Dirac spinor at time $t^5 = 0$, except that the polarization sums denoted by “o”, should run over two, instead of four values. The necessity to reproduce the continued $3 + 1$ dimensional Minkowski propagator, however, can be used to rule out directly replacing ψ_{E} and $\chi_{\text{E}}^{\dagger}$ by their zero time, five dimensional counterparts.

⁵To see this, take a basis in which $\gamma_{\text{E}}^5 = \text{diag}(1, 1, -1, -1)$ and the other Dirac matrices are off-diagonal. Multiplying by $-\gamma_{\text{E}}^5$ and tracing yields $U_k^R \equiv 0$ ($R = 1, \dots, 4$).

In this light, we consider the more general ansatz

$$\psi_E(x) = \int \frac{d^4k}{\sqrt{(2\pi)^4}} \left\{ \mathcal{B}_k e^{ikx} + \mathcal{D}_k^* e^{-ikx} \right\}, \quad (41)$$

from which it follows that

$$-\psi_E^\dagger(x) \gamma_E^5 = - \int \frac{d^4k}{\sqrt{(2\pi)^4}} \left\{ \mathcal{B}_k^\dagger \gamma_E^5 e^{-ikx} + \mathcal{D}_k^\top \gamma_E^5 e^{ikx} \right\}, \quad (42)$$

where we have replaced the combinations $U_k \circ \mathcal{B}_k$ and $D_k^\dagger \circ V_{-k}$ by the operator-valued four-spinors \mathcal{B}_k and \mathcal{D}_k^* , respectively. We still require that $\mathcal{B}_k |0\rangle_E = 0 = \mathcal{D}_k |0\rangle_E$.

By virtue of the rotation on spinor indices, the unitary involution Θ is now defined on spinors in the same way as for bosons

$$\Theta \psi_E(x) \Theta^{-1} = \psi_E(\Theta x); \quad \Theta \psi_E^\dagger(x) \Theta^{-1} = \psi_E^\dagger(\Theta x), \quad (43)$$

which in turn defines the action of Θ on the spinor modes

$$\Theta \mathcal{B}_k^\dagger \Theta^{-1} = \mathcal{B}_{\Theta k}^\dagger, \quad (44)$$

$$\Theta \mathcal{D}_k^\top \Theta^{-1} = \mathcal{D}_{\Theta k}^\top. \quad (45)$$

Clearly this action is involutive. Euclidean ultralocality,

$$0 = \{\psi_E(x), -\psi_E(y)^\dagger \gamma_E^5\}, \quad (46)$$

is satisfied by requiring

$$\{\mathcal{B}_k, \mathcal{B}_{k'}^\dagger\} = -\{\mathcal{D}_{-k}^*, \mathcal{D}_{-k'}^\top\}. \quad (47)$$

The correct two-point function as in (39) is ensured by the following anti-commutation relations for the “spinor modes”

$$-\{\mathcal{B}_k, \mathcal{B}_{k'}^\dagger \gamma_E^5\} = \frac{-ik_k^E + m}{\Omega_k^2} \delta^4(k - k'). \quad (48)$$

The matrix $\left(\frac{-ik_k^E + m}{\Omega_k^2} \right) (-\gamma_E^5)$ is hermitean with eigenvalues $\pm \Omega_k$ and may be diagonalized (in a basis for the Dirac matrices in which γ_E^5 is diagonal) by defining

$$\mathcal{B}_k = \frac{1}{\sqrt{\Omega_k}} \mathcal{U}_k \tilde{\mathcal{B}}_k \quad (49)$$

where

$$\mathcal{U}_k = \frac{\Omega_k - m + i k^E}{\sqrt{2(\Omega_k - m)\Omega_k}} = (\mathcal{U}_k^{-1})^\dagger = (\mathcal{U}_{-k})^{-1}. \quad (50)$$

The combination $\frac{1}{\sqrt{\Omega_k}}\mathcal{U}_k$ is the analogue of the OS spinor wave function U_k . In this basis the mode relations now read

$$\{\tilde{\mathcal{B}}_k, \tilde{\mathcal{B}}_{k'}^\dagger\} = -\gamma_E^5 \delta^4(k - k'). \quad (51)$$

That the Euclidean Fock space now contains negative norm states causes no problems since we only require positive norm in the Minkowski Hilbert space, which is assured since, by construction, our fields satisfy the same axioms as those of the OS proposal. (It is amusing to note that although we still double the number of mode operators, we find that half of them have negative norm in the Euclidean Fock space).

We may “diagonalize” the \mathcal{D}_k modes in a similar fashion so that the final result for our Euclidean fields reads

$$\psi_E(x) = \int \frac{d^4k}{\sqrt{(2\pi)^4\Omega_k}} \left\{ U_k \tilde{\mathcal{B}}_k e^{ikx} + U_{-k} \tilde{\mathcal{D}}_k^* e^{-ikx} \right\}, \quad (52)$$

and

$$\chi_E^\dagger(x) = -\psi_E^\dagger(x) \gamma_E^5 = -\Theta^{-1} \psi_E^\dagger(\Theta x) \gamma_E^5 \Theta. \quad (53)$$

The modes satisfy commutation relations

$$\{\tilde{\mathcal{B}}_k, \tilde{\mathcal{B}}_{k'}^\dagger\} = -\gamma_E^5 \delta^4(k - k') = -\{\tilde{\mathcal{D}}_{-k}^*, \tilde{\mathcal{D}}_{-k'}^\top\}, \quad (54)$$

By construction these fields anticommute at all points in Euclidean space

$$\{\psi_E(x), \chi_E^\dagger(y)\} \equiv -\{\psi_E(x), \psi_E^\dagger(y) \gamma_E^5\} = 0 \quad (55)$$

and possess the desired two-point function,

$${}_E\langle 0 | \psi_E(x) \psi_E^\dagger(y) (-\gamma_E^5) | 0 \rangle_E = \int \frac{d^4k}{(2\pi)^4} \frac{-i k^E + m}{k_E^2 + m^2} e^{ik_E(x-y)_E}. \quad (56)$$

which is consistent with hermiticity. We have now reproduced the building blocks ((56), (54), (43) and (38)) of the OS construction and the rest of their proposal may now be inherited unaltered except for the replacement everywhere of the field χ_E^\dagger by $-\psi_E^\dagger \gamma_E^5$ and the extra rotation of spinor indices performed by the new OS–Wick map.

4 Conclusion

In this note we have presented a generalization of the canonical work of Osterwalder and Schrader for Dirac spinors which fuses their approach with the Schwinger–Zumino [4][5] approach in which hermiticity is maintained although we still found it necessary to double the set of Euclidean fermionic mode operators. This was achieved by considering a new Wick rotation for fermions under which spinor indices also rotate. We would also suggest, that our generalization adds to the formal simplicity of the OS construction.

Finally, one may wonder what happens in the case of Majorana or Weyl spinors. In [1] we found that for Majorana and Weyl spinors in four dimensions that the requirement of hermiticity must be dropped. The extension of the work of OS to Majorana spinors was given by Nicolai [6] who noted that although there existed no consistent reality condition for Majorana spinors in four dimensional Euclidean space, one could nonetheless define a symplectic reality condition on the mode operators B_k and D_k following which the OS proposal may also be simply inherited. It is not a difficult matter to apply the generalization we have given above also to Nicolai’s work, and although one cannot regain hermiticity, the same formal algebraic simplifications as above occur. One may then even study $N = 1$ supersymmetric systems. As usual real bose fields undergo complex supersymmetry transformations in Euclidean space since the canonical supersymmetry charge Q no longer satisfies any reality condition. Such considerations may also be formulated in superspace (see also [7]).

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